Finding Maxima and Minima of functions with more than 2 variables

When a function has one variable we know how to find the maxima and minima of the function by differentiating and equating to zero to find the points.

But when a function has more than two variables, we use partial differentiation to find the maxima and minima.

1. $f(x, y) = x^3 + 3xy^2 + 2xy$ subject to the condition x + y = 4

Sol: The local maximum and minimum of f(x,y) subject to the constraint g(x,y)=0 correspond to the stationary points of $L(x, y, \lambda) = f(x, y) - \lambda. g(x, y)$

where λ is Lagrange multiplier.

We have $L(x, y, \lambda) = x^3 + 3xy^2 + 2xy - \lambda. (x + y - 4)$

Now $\frac{\partial L}{\partial x} = 3x^2 + 3y^2 + 2y - \lambda$

$\frac{\lambda L}{\lambda y} = 6xy + 2x - \lambda$

$\frac{\partial L}{\partial y} = 6xy + 2x - \lambda$

$\frac{\partial L}{\partial \lambda} = -(x + y - 4)$

Note: in doing partial differentiation, except the independent variable everything is considered a constant. For example, when we do differentiation w.r.t x then, except x all others should be considered constant.

Also $\frac{d}{dx}(x^n) = n. x^{n-1}$

Putting $\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial \lambda} = 0$

$3x^2 + 3y^2 + 2y = \lambda$ ------(1)

$6xy + 2x = \lambda$ -----(2)

x + y - 4 = 0 ------------(3)

From equations (1) and (2) we get $3x^2 + 3y^2 + 2y = 6xy + 2x$ --------(4)

Putting y = - x + 4 in equation (4)

we get $3x^2 + 3(-x + 4)^2 + 2(-x + 4) = 6x(-x + 4) + 2x$

$3x^2 + 3(x^2 - 8x + 16) - 2x + 8 = -6x^2 + 24x + 2x$

$12x^2 - 52x + 56 = 0$

$\Rightarrow 3x^2 - 13x + 14 = 0$

$\Rightarrow (3x - 7)(x - 2) = 0$

$\Rightarrow x = 2, \frac{7}{3}$

For x = 2, from equation (3) we get y = 2 and F(x,y)) = 40

and for $x = \frac{7}{3}$, $y = \frac{5}{3}$ and F(x,y) = $39\frac{25}{27}$

Alternatively: By substituting y = x - 4 in the equation f(x, y) = $x^3 + 3xy^2 + 2xy$

we get, F(x, 4-x) = $x^3 + 3x(4-x)^2 + 2x(4-x)$

F(x) = $x^3 + 3x(16 - 8x + x^2) + 2x(4-x)$

F(x) = $4x^3 - 26x^2 + 56x$

Differentiation F with respect to x we get, $F^1(x) = 12x^2 - 52x + 56$

Solving like above we get the values of $\Rightarrow x = 2, \frac{7}{3}$

2. Find the point on the line 3x + 2y = 5 that is closest to the point (3,1)

Sol: The distance between a general point (x,y) and the point (3, 1) is $\sqrt{(x-3)^2 + (y-1)^2}$

We want to find the minimum value of this distance subject to the constraint 3x + 2y = 5. Infact we have to minimize the square of the distance, and so we minimize f(x, y) = $(x-3)^2 + (y-1)^2$

subject to the given constraint.

$L(x, y, \lambda) = (x-3)^2 + (y-1)^2 - \lambda(3x + 2y - 5)$

$\frac{\partial L}{\partial x} = 2(x-3) + 3\lambda$

$\frac{\partial L}{\partial y} = 2(y-1) + 2\lambda$

$\frac{\partial L}{\partial \lambda} = -3x - 2y + 5$

Putting $\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial \lambda} = 0$

$2(x-3) + 3\lambda = 0$ --------(1)

$2(y-1) + 2\lambda = 0$ -------(2)

3x + 2y = 5 ------------(3)

Multiplying (1) by 2 and (2) by 3 will give

$4(x-3) + 6\lambda = 0$

$6(y-1) + 6\lambda = 0$

So 4(x-3) = 6(y-1) ⇒ 2x - 3y = 3 -------(4)

Multiplying equation (3) by 3 and (4) by 2, gives

9x + 6y = 15

4x - 6y = 6

Solving we get x = $\frac{21}{13}$ and y = $\frac{1}{13}$

Thus the point $(\frac{21}{13}, \frac{1}{13})$ is on the given line and closest to (3,1)